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String model in $D=1+3$ dimensions: the non-standard approach to Hamiltonian dynamics and quantization

S V Talalov

Department of Theoretical Physics, Pedagogical Institute, 445859 Togliatti, Russia

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Abstract. The open spinning string is investigated in Minkowski space $E_{1,3}$. By means of a reduction to the Wess–Zumino–Witten–Novikov (WZWN) model a new Hamiltonian formalism is constructed. The Poisson bracket structure of the theory is given in terms of the algebra $[(sl(2, C)\otimes[t^{-1}, t])\oplus C_z]\otimes P$, where P is the Poincaré algebra. The covariant quantization is fulfilled in $D=1+3$ dimensions with help of 'bosonization' methods. The resulting theory is a combination of the theory of a free fermionic field in two dimensions and the theory of a free particle in Minkowski space. Non-triviality is conditioned by the presence of a finite number of constraints. The formula for the mass spectrum is discussed.

1. Introduction

The classical dynamics of the string as a curve in D-dimensional spacetime was investigated many years ago in various aspects. Because of the natural geometrical interpretation of the string action S the Lagrangian formalism in unambiguous here. The situation with the Hamiltonian theory is more complicated. Indeed, it is known that standard Hamiltonian variables $(\mu=0, 1, \ldots, D-1)$

$$X_\mu(\xi^0, \xi^1) \qquad \Pi_\mu(\xi^0, \xi^1)=\frac{\delta S}{\delta \dot{X}_\mu(\xi^0, \xi^1)}, \ldots \tag{1}$$

where (ξ^0, ξ^1) is some parametrization of the string world sheet $\{X_\mu(\xi^0, \xi^1)\}$, allows construction of the covariant quantum theory only for $D=26$ $(D=10)$ (Scherk 1975, Green *et al* 1987). Because of the obvious contradiction with the contemporary experimental data on the spacetime properties the following alternative theoretical developments exist. First, the superfluous dimensions must be compacted into a Planck-size manifold in some way. The second opportunity which is the starting point of our investigation is connected with the following theoretical statement (Dirac 1950, Berezin 1975): any fixed classical dynamical system can possess various Hamiltonian structures which are different and non-equivalent to the standard one (1). The classical system can be fixed by means of equations of motion, for example. In accordance with the works of Dirac (1950) and Berezin (1975), each of these (non-standard) Hamiltonian structures can be considered as the primary point of the corresponding classical theory and be subjected to the quantization procedure. Of course, the results are different for the various Hamiltonian variables, but the experiment is the only criterion here! We

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state that the Poincaré algebra P must be correct after quantization in $D=1+3$ dimensions. This demand is a 'selection rule' for the possible canonical Hamiltonian variables. Indeed, different choices lead to different formulae for the generators P^{μ} and $M^{\mu\nu}$ of the spacetime transformations. In general, we have various anomalous terms in the quantum algebra P. However, because of the wide choice of variables there is no reason to always expect the presence of these terms. (It should be mentioned that we consider only the anomaly in Poincaré commutators.)

Various approaches exist in connection with quantum string theory in Minkowski space (Rohrlich 1976, Polyakov 1981, Pron'ko 1985, Lunev 1990). One of the possible models which conserves the covariancy in four dimensions is assumed in this paper. Some previous investigations have been made (Talalov, 1990a, b), and we want to use some of those results. The object of our studies is the open string $X^{\mu}=X^{\mu}(\xi^1)$ where $\xi^1\in[0,\pi]$ and the Majorana spinor field $\Psi_i^A=\Psi_i^A(\xi^1)$, which is the spinor-valued function defined on the world sheet. The index A is a four-dimensional spinor index in the space $E_{1,3}$ and the index i is a two-dimensional spinor index in the corresponding tangent plane. Note that components Ψ_i^A are complex (non-Grassmann!) numbers. The ξ^0 dynamics is defined by means of relativistic invariant action,

$$S=-\tfrac{1}{2}\int \mathrm{d}\xi^0\,\mathrm{d}\xi^1(\partial_a X_\mu \partial^a X^\mu - \mathrm{i}\bar{\Psi}^A\gamma^b\partial_b\Psi^A) \tag{2}$$

where the two-dimensional γ-matrices $\gamma^0\equiv\sigma_1$, $\gamma^1\equiv\mathrm{i}\sigma_2$ and $\partial_a=\partial/\partial\xi^a$. It is well known that the action (2) can be deduced consistently from the action for superstring theory (Green *et al* 1987) by fixing its initial symmetries. For example, the full reparametrization invariance was broken by the choice of orthonormal parametrization of the string world sheet, so that

$$(\partial_\pm X_\mu)^2=0 \tag{3}$$

where $\partial_\pm\equiv\partial/\partial\xi_\pm$ and $\xi_\pm\equiv\xi^1\pm\xi^0$ are the cone parameters. Of course, the conditions (3) are fulfilled in our work.

However, some starting positions of our model are quite different from ones for the superstring theory. The ordinary supersymmetry transformations which connect the variables X_μ and Ψ_a will not be considered in the suggested approach. These variables become complicated functionals on fundamental Hamiltonian variables which will be introduced below. *A priori*, the only demand here is the relativistic invariance.

In accordance with the above discussions, the following will be emphasized here. The action (2) serves two aims only. First, we want to have the equations of motion

$$\partial_-\partial_+X_\mu=0 \qquad \partial_\mp\Psi_\pm=0 \tag{4}$$

where $\Psi_+\equiv\Psi_1$, $\Psi_-\equiv\Psi_2$, with the corresponding boundary conditions

$$\begin{aligned} &X'_\mu|_{\xi^1=0}=X'_\mu|_{\xi^1=\pi}=0\\ &\Psi_+|_{\xi^1=0}=\Psi_-|_{\xi^1=0} \qquad \Psi_+|_{\xi^1=\pi}=\Psi_-|_{\xi^1=\pi}\end{aligned} \tag{5}$$

and second we want to have the explicit formulae for the Nöether invariants P^{μ} and $M^{\mu\nu}$. The functions (1) and $\Psi_\pm$, $\Psi_\pm^*$ are not canonical Hamiltonian variables in our model. The Poisson brackets for these functions are complicated.

2. Gauge fixing and reduction to the WZWN model

As is well known (Green *et al* 1987), the light-cone gauge is one of the necessary conditions which must be imposed for the passage to action (2) from the Green–Schwarz action. In accordance with the above discussion we start from the action (2). That is why we must choose some gauge which is additional for the dynamical equations in our model. Our conditions are as follows:

$$\bar{\Psi}_\pm \Gamma^\mu \Psi_\pm \partial_\pm X_\mu = \mp \tfrac{1}{2}. \tag{6}$$

We have some natural generalization of the light-cone gauge. Indeed, the functions $\Psi_\pm = \text{const}$ are the particular solutions of the dynamical equations for the spinors $\Psi_\pm(\xi)$. In this case the vector $n^\mu = \bar{\Psi}_\pm \Gamma^\mu \Psi_\pm$ is the constant light-like vector and conditions (6) are reduced to the form

$$\partial_\pm X_\mu n^\mu \neq 0. \tag{7}$$

It is known that the standard light-cone gauge can be imposed iff the inequality (7) occurs for $n_\mu = (1, 0, 0, 1)$.

Because of the close connection with string theory, two-dimensional conformal models are subject to intensive study (e.g. see Belavin *et al* 1984, Alvarez-Gaume *et al* 1987, Sedrakyan and Stora 1987). Our approach is based on the following construction. Let us consider the system of tensors $(Y^\mu, G^{\mu\nu})$, where $Y^\mu = \bar{\Psi}\Gamma^\mu\Psi$, and $G^{\mu\nu} = (i/2)\bar{\Psi}(\Gamma^\mu\Gamma^\nu - \Gamma^\nu\Gamma^\mu)\Psi$. As is well known, the following one-to-one correspondence exists (Penrose and Rindler 1984): $\Psi \Leftrightarrow (Y^\mu, G^{\mu\nu})$. Next, we construct the pair of bases $e_{\mu+}$ and $e_{\mu-}$ which are defined at each point of the string world sheet:

$$(e_{0\pm})^\mu = Y^\mu_\pm \pm \partial_\pm X^\mu \qquad (e_{3\pm})^\mu = \pm Y^\mu_\pm - \partial_\pm X^\mu$$
$$(e_{1\pm})^\mu = \mp 2 G^{\mu\nu}_\pm \partial_\pm X_\nu \qquad (e_2)^\mu = \varepsilon^{\mu\nu\lambda\rho}(e_3)_\nu (e_0)_\lambda (e_1)_\rho .$$

Because of conditions (3) and (6) and the properties of tensors Y_μ and $G_{\mu\nu}$ (Zhelnorovich 1979), these bases are orthonormal. Let us define the pair of vector matrices

$$\mathbf{E}_\pm = e_0 1 - \sum_{i=1}^{3} e_{i\pm}\sigma_i$$

and introduce the matrix $K = K(\xi^0, \xi^1)$ which transforms from E_+ to E_-:

$$\mathbf{E}_- = K \mathbf{E}_+ K^\dagger. \tag{8}$$

As a consequence of the equations of motions for the variables X_μ and $\Psi_\pm$, we have

$$\partial_\pm \mathbf{E}_\mp = 0.$$

Therefore, the following equation for the $SL(2, C)$-valued field $K = K(\xi^0, \xi^1)$ is obtained:

$$\partial_-(K^{-1}\partial_+ K) = 0. \tag{9}$$

This is a special case of the general equation for the WZWN model

$$a\partial_-(K^{-1}\partial_+ K) + b\partial_+(K^{-1}\partial_- K) = 0.$$

Define now the left and right currents:

$$Q_- = -(\partial_- K)K^{-1} \qquad Q_+ = K^{-1}(\partial_+ K). \tag{10}$$

The following statement is true: *the periodical functions*

$$Q(\xi)=\begin{cases}Q(\xi) & \text{if } \xi\in[0,\pi]\\ -\sigma_1 Q_-(-\xi)\sigma_1 & \text{if } \xi\in[-\pi,0]\end{cases}$$

can be constructed correctly with the help of the boundary conditions (5).

The proof is quite simple if we rewrite the boundary conditions (5) in the form

$$\mathbf{E}_-|_{\xi^1=0,\pi}=\sigma_1\mathbf{E}_+|_{\xi^1=0,\pi}\sigma_1. \tag{11}$$

Because of equalities (11) which take place for every ξ^0, matrices $\mathbf{E}_\pm$ can be continued periodically. As a consequence, the periodical $sl(2, C)$-valued function $Q(\xi)$ may be constructed.

Let us consider the auxiliary linear system

$$T'(\xi)+Q(\xi)T(\xi)=0. \tag{12}$$

As a consequence of formulae (8)–(10), the vector matrices $\mathbf{E}_\pm$ are reconstructed explicitly through the matrix $T(\xi)$—the matrix of solutions of the system (11):

$$\begin{aligned}\mathbf{E}_+(\xi_+)&=T(\xi_+)\mathbf{E}_0T^\dagger(\xi_+)\\ \mathbf{E}_-(\xi_-)&=\sigma_1T(-\xi_-)\mathbf{E}_0T^\dagger(\xi_-)\sigma_1\end{aligned} \tag{13}$$

where the vector matrix $\mathbf{E}_0$ is some constant basic matrix. It is clear that the original string variables $\partial_\pm X_\mu$ and $\Psi_\pm$ can be reconstructed through the elements of the matrices $\mathbf{E}_\pm$.

3. Poisson bracket structure and world sheet geometry

Let us define the current $j_a(\xi)$ by means of the decomposition

$$Q(\xi)=\frac{i}{2}\sum j_a(\xi)\sigma_a.$$

The general Hamiltonian structure of the two-dimensional chiral field theories (Takhtadjan and Faddeev 1986) allow us to write the following Poisson brackets for the variables $j_a(\xi)$:

$$\{j_a(\xi),j_b(\eta)\}_0=-2\delta_{ab}\delta'(\xi-\eta)-2\varepsilon_{abc}j_c(\xi)\delta(\xi-\eta) \tag{14}$$

where $\delta(x)=\Sigma_n\,e^{inx}$. As a consequence of the boundary conditions (11) and formulae (13), we have to impose the following 'constraints' on the Hamiltonian variables j_a:

$$\mathbf{M}(j_a)=1 \tag{15}$$

where $\mathbf{M}=\mathbf{M}(j_a)$ is the monodromy matrix of the system (12). The remarkable fact is that the matrix $\mathbf{M}$ annuls all brackets on the surface of the constraints (15) because

$$\{Q(\xi)\otimes,\mathbf{M}\}_0=[1\otimes\mathbf{M},\mathbf{C}(\xi)] \tag{16}$$

for some 4×4 matrix $\mathbf{C}$, which is defined from the concrete representation for $\mathbf{M}$. The proof and a discussion can be found in Talalov 1990a, b).

The relevance of the brackets (14) in our model is confirmed by means of the following geometrical arguments. Let us fulfil the Gauss decomposition for the matrix $K^{-1}(\xi^0, \xi^1)$:

$$K^{-1} = \begin{pmatrix} 1 & 0 \\ -\alpha_+ & 1 \end{pmatrix} \begin{pmatrix} \exp(-\phi/2) & 0 \\ 0 & \exp\phi/2 \end{pmatrix} \begin{pmatrix} 1 & \alpha_- \\ 0 & 1 \end{pmatrix}. \tag{17}$$

Next, we introduce the pair of functions $\rho_\pm = (\partial_\pm \alpha_\mp)\, e^{-\phi}$. By analogy to the work of Talalov (1989), where the case $D = 1+2$ dimensions was investigated, the following statement can be proved: *the first* (I) *and second* (II_1, II_2) *quadratic forms of the string world sheet are written by means of the formulae*

$$I = -\tfrac{1}{2} e^{-\mathrm{Re}\,\phi}\, d\xi_+\, d\xi_-$$

$$II_1 - iII_2 = e^{i(\mathrm{Im}\,\phi/2)}[\rho_+\, d\xi_+^2 + \rho_-\, d\xi_-^2].$$

In accordance with the definitions (10) and (17) the components of current $j_a(\xi)$ are the local functionals from the variables $\phi(\xi)$, $\pi(\xi) \equiv \phi(\xi)$, $\rho_\pm(\xi)$, $\alpha_\pm(\xi)$. The following fact can be stated directly: *the brackets* (14) *are the consequence of the fundamental brackets for variables* ϕ, π, $\alpha_\pm$, $\rho_\pm$:

$$\{\pi(\xi), \phi(\eta)\}_0 = 4\delta(\xi - \eta) \qquad \{\rho_\pm(\xi), \alpha_\pm(\eta)\}_0 = \pm\delta(\xi - \eta) \tag{18}$$

(*other brackets vanish*). As is well known, the Gauss decomposition (17) may be fulfilled everywhere except at some points on the (ξ^0, ξ^1) plane where the following equality is true:

$$(K^{-1})_{11} = \exp\left(-\frac{\phi}{2}\right) = 0.$$

This means that we must consider the singular functions ϕ, $\alpha_\pm$ too. From the geometrical viewpoint this situation corresponds to the strings which are cuspidal curves. This case is of course interesting. Because of the presence of singularities, definitions of the 'small' variations $\delta\phi$, $\delta\alpha_\pm$ are ambiguous. That is why we cannot use the functions ϕ, π, $\alpha_\pm$, $\rho_\pm$ for parametrization of the phase space in our model. To avoid a lot of problems we must deal with the (regular) current $j_a(\xi)$ and the brackets (14).

In accordance with the rule of reconstruction of the variables $\partial_\pm X^\mu(\xi^0, \xi^1)$ and $\Psi_\pm(\xi_\pm)$, the set $\{j_a(\xi)\}$ is insufficient. Indeed, this set must be completed by the finite number of constants $\{A_{ij}\}$. These constants must fix the matrix $T(\xi)$, because $T(\xi)$ can be subjected to the following transformations:

$$T(\xi) \to \tilde{T}(\xi) = T(\xi)B \tag{19}$$

where the constant matrix $B \in GL(2, C)$. Moreover, four constants Z^μ are needed for the reconstruction of variables X^μ from the derivatives $\partial_\pm X^\mu$. The quantities Z^μ and A_{ij} are additional independent Hamiltonian variables. The set $\{A\}$ can be chosen in various ways, for example

$$T(0) = A \qquad \text{or} \qquad \int_0^{2\pi} T(x)\, dx = A. \tag{20}$$

As follows from formula (13), the transformations (19) are the Lorentz transformations and dilatations of the initial Minkowski space $E_{1,3}$. That is why the variables $\{A_{ij}\}$ are not relativistic invariant, unlike the current $j_a(\xi)$, which is invariant. Obviously, the constants Z^μ are changed by Poincaré translations. In accordance with the above discussion the Poisson brackets (14) must be completed by some brackets $\{A_{ij}, A_{mn}\}$,

$\{Z_\mu, Z_\nu\}, \ldots$. Thus, the Poisson brackets of the two arbitrary functionals G, F in our theory must be defined, in general, by the formula

$$\{G, F\} = \int_0^{2\pi} \frac{\delta G}{\delta j_a(\xi)} \frac{\delta F}{\delta j_b(\eta)} \{j_a(\xi), j_b(\eta)\}_0 \,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$+\frac{\partial F}{\partial A_{ij}} \frac{\partial G}{\partial A_{kl}} V_{ijkl} + \left(\frac{\partial F}{\partial A_{ij}} \frac{\partial G}{\partial Z_\mu} - (F \leftrightarrow G)\right) S^\mu_{ij}$$

where the brackets $\{j_a, j_b\}_0$ have been defined by formula (14) and V_{ijkl}, S^μ_{ij} are some appropriate constants. Thus we have the number of symplectic structures which differ from each other by some 'boundary' terms. This situation is well known from the Hamiltonian theory of the inverse scattering transform for some nonlinear equations (e.g. see Arkad'ev *et al* 1988). We have to emphasize the following point here. The various definitions (20) lead to the non-equivalent Poisson bracket structure, in general. We must choose those variables which conserve the relativistic covariancy of our theory after quantization.

Let us consider the energy momentum P_μ and the angular momenta $M_{\mu\nu}$, which are functionals of the variables X_μ and $\Psi_\pm$ in accordance with the Nöether theorem. Then for the matrices

$$\hat{P} = \sigma_\mu P^\mu \qquad \hat{M} = \sigma_\mu \otimes \sigma_\nu M^{\mu\nu} \qquad \hat{Z} = \sigma_\mu Z^\mu$$

where $\sigma^0 \equiv 1$, we have (Talalov 1990a, b)

$$\hat{P} = \tfrac{1}{2} \int_0^{2\pi} T^\dagger(x)(1+\sigma_3)T(x)\,\mathrm{d}x \tag{21}$$

$$\hat{M} = \hat{Z} \otimes \hat{P} - \hat{P} \otimes \hat{Z}$$

$$+ \int_0^{2\pi} (T^\dagger(x) \otimes T^\dagger(y)) W(x-y)(T(x) \otimes T(y))\,\mathrm{d}x\,\mathrm{d}y$$

where $W(x)$ is some matrix-value distribution.

The idea is that we can fix the matrix solution $T(\xi)$ and integration constants Z_μ by fixing the variables P_μ and $M_{\mu\nu}$. Indeed, let P_μ and $M_{\mu\nu}$ be arbitrary constant tensors with the properties $(P_\mu)^2 > 0$ and $M_{\mu\nu} = -M_{\nu\mu}$. Let $P_{(0)\mu}$ and $M_{(0)\mu\nu}$ be invariants which are calculated in accordance with formulae (21) for $\hat{Z} \equiv 0$ and $T(x) = T_0(x)$ where $T_0(x)$ is the matrix solution of the system (12) which is fixed by the condition

$$T_0(x)|_{x=0} = 1.$$

Then for every fixed current j_a the constants P_μ, $M_{\mu\nu}$ correspond to the unique matrix solution $T(\xi)$ of the system (12) and the unique set of integration constants $\{Z_\mu\}$ if the following additional condition is fulfilled (Talalov, 190a, b):

$$(P_\mu P^\mu)^2 (S_{(0)\nu} S^\nu_{(0)}) = (S_\rho S^\rho)(P_{(0)\tau} P^\tau_{(0)})^2 \tag{22}$$

where the Luban'ski–Pauli vector $S_\mu = -(1/2\sqrt{P_\omega P^\omega})\varepsilon_{\mu\nu\lambda\sigma} M^{\nu\lambda} P^\sigma$ is introduced and the vector $S_{(0)\mu}$ must be written through the quantities $P_{(0)\mu}$ and $M_{(0)\mu\nu}$, correspondingly. The quantities with the index (0) are the single-valued functionals from the currents j_a. Note that the different powers of the multipliers in equation (22) are conditioned by the different properties of the vectors $P_{(0)\mu}$, $S_{(0)\mu}$ under the dilatations. Because of condition (22), two pairs of orthogonal vectors (P_μ, S_μ) and $(P_{(0)\mu}, S_{(0)\mu})$ can be transformed into each other by the Lorentz transformation and dilatation (19).

Thus the independent Hamiltonian variables which must complete the set $\{j_a(\xi)\}$ are the energy momentum P_μ and the boosts and angular momenta $M_{\mu\nu}$. We postulate the following Poisson bracket structure for our theory:

$$\{F, G\}=4\pi \int_0^{2\pi} \left(\frac{\delta F}{\delta j_a(\xi)}\right)' \frac{\delta G}{\delta j_a(\xi)} \mathrm{d}\xi - 4\pi\, \varepsilon_{abc} \int_0^{2\pi} \frac{\delta F}{\delta j_a(\xi)} \frac{\delta G}{\delta j_b(\xi)} j_c(\xi)\, \mathrm{d}\xi$$
$$+\frac{\partial F}{\partial M_{\alpha\beta}} \frac{\partial G}{\partial M_{\gamma\delta}} (g_{\alpha\delta} M_{\beta\gamma} + g_{\beta\gamma} M_{\alpha\delta} - g_{\alpha\gamma} M_{\beta\delta} - g_{\beta\delta} M_{\alpha\gamma})$$
$$+\left(\frac{\partial F}{\partial M_{\alpha\beta}} \frac{\partial G}{\partial P_\gamma} - \frac{\partial F}{\partial P_\gamma} \frac{\partial G}{\partial M_{\alpha\beta}}\right)(g_{\beta\gamma} P_\alpha - g_{\alpha\gamma} P_\beta)$$

where $F=F(j_a(\xi);\, P_\mu, M_{\mu\nu})$ and $G=G(\ldots)$ are some functionals. Obviously, these brackets are antisymmetrical. The Jackobi identity is the next important demand here. It may be verified directly for the functionals $F, \ldots,$ such that the variational derivatives $\delta F \backslash \delta j_a(\xi), \ldots,$ are 2π-periodical functions. It must be emphasized that this condition is trivial on the surface of constraints (15) only. Thus the brackets of the variables P_μ and $M_{\mu\nu}$ are

$$\begin{aligned} &\{P_\alpha, P_\beta\}=0 \qquad \{M_{\alpha\beta}, P_\gamma\}=g_{\beta\gamma}P_\alpha - g_{\alpha\gamma}P_\beta \\ &\{M_{\alpha\beta}, M_{\gamma\delta}\}=g_{\alpha\delta}M_{\beta\gamma}+g_{\beta\gamma}M_{\alpha\delta}-g_{\alpha\gamma}M_{\beta\delta}-g_{\beta\delta}M_{\alpha\gamma} \\ &\{j_a, P_\alpha\}=\{j_a, M_{\alpha\beta}\}=0. \end{aligned} \tag{23}$$

Because the variables P_μ and $M_{\mu\nu}$ are independent Hamiltonian variables, the anomalous terms do not appear in formulae (23) after quantization. The correspondence

$$(X_\mu, \Psi_\pm) \leftrightarrow (j_a(\xi);\, P_\mu, M_{\mu\nu})$$

takes place on the surface of constraints (15) and (22). In accordance with (16) these constraints are the first type constraints in Dirac terminology.

The Hamiltonian

$$H_0=\tfrac{1}{2}\int_0^{2\pi} \sum j_a^2(\xi)\, \mathrm{d}\xi \tag{24}$$

leads to the following ξ^0 dynamics:

$$\begin{aligned} &\{H_0, \partial_0 X_\mu\}=\partial_0^2 X_\mu \\ &\{H_0, \Psi_\pm\}=\partial_0 \Psi_\pm. \end{aligned} \tag{25}$$

The brackets of H_0 with constraints (15) and (22) vanish on the constraint surface. The pair of independent canonical variables p, q, where $\{p, q\}=1$, must be added for the correct integration of equation (25). Indeed, the Hamiltonian

$$H=H_0+p$$

leads to the relation

$$\{H, X_\mu\}=\partial_0 X_\mu$$

for

$$X_\mu(\xi^0, \xi^1)=Z_\mu+P_\mu(\xi^0+q)+\ \ldots\ 2\pi\text{-periodical terms.}$$

In the last formula $\{H_0, Z_\mu\}=0$ because $Z_\mu=Z_\mu(P, M; P_{(0)}, M_{(0)})$ in accordance with the above discussion. We will not take into account the variables p, q in this article. The constraints $p=0$, $q=0$ are solved trivially with the help of the corresponding Dirac brackets construction.

The field $K(\xi^0, \xi^1)$ will be reduced to the $SU(2)$-valued field in the next section. That is why the current $j_a(\xi)$ is real and the brackets $\{j_a, j_b^*\}$ are not needed by us.

4. Conformal properties and reduction to the real currents

Let us rewrite equation (9) in terms of the (complex) functions $\phi(\xi^0, \xi^1)$, $\alpha_\pm(\xi^0, \xi^1)$, which are introduced by means of decomposition (17), and functions $\rho_\pm(\xi^0, \xi^1)$ which are introduced above. The direct calculations gives the following result:

$$
\begin{aligned}
&-\tfrac{1}{2}\partial_+\partial_-\phi+\rho_+\rho_-\exp\phi=0\\
&\partial_\pm\rho_\mp=0\\
&\partial_\pm\alpha_\mp-\rho_\pm\exp\phi=0.
\end{aligned}
\tag{26}
$$

This system was considered in the work of Pogrebkov and Talalov (1987) for the real functions $\phi, \alpha_\pm, \rho_\pm\geqslant 0$ as the field model in two-dimensional spacetime. This model is the product of the well-known Thirring and Liouville models if the two-dimensional spinor field with components $\Theta_\pm=\sqrt{\rho_\pm}\exp(\pm 4i\alpha_\pm)$ are defined.

The wide group of conformal symmetry of the system (26) was also investigated in the paper by Pogrebkov and Talalov (1987). In our (complex) case the conformal properties are similar. Indeed, consider the functions $f_\pm$, $g_\pm$ and $A_\pm$ where $A'_-A'_+\neq 0$ are arbitrary differentiable functions. Then the transformation

$$
(\phi, \rho_\pm, \alpha_\pm)\to(\tilde{\phi}, \tilde{\rho}_\pm, \tilde{\alpha}_\pm)
\tag{27}
$$

where

$$
\begin{aligned}
&\tilde{\phi}(\xi_+, \xi_-)=\phi(A_+(\xi_+), A_-(\xi_-))+f_+(\xi_+)+f_-(\xi_-)\\
&\tilde{\rho}_\pm(\xi_\pm)=\rho(A_\pm(\xi_\pm))A'_\pm(\xi_\pm)\exp(-f_\pm(\xi_\pm))\\
&\tilde{\alpha}_\pm(\xi_+, \xi_-)=\alpha_\pm(A_+(\xi_+), A_-(\xi_-))\exp(f_\pm(\xi_\pm))+g_\pm(\xi_\pm)
\end{aligned}
$$

gives other solutions of the system (26). The generalization of the well-known conformal invariance of the Liouville equation $\partial_+\partial_-\varphi+\exp\varphi=0$ is obtained. Note that the functions $\rho_\pm$ are dynamical variables in our model and cannot be exempted from our consideration. From the geometrical viewpoint, the following complex differential form on the string world sheet is invariant under the transformation (27):

$$
\mathrm{d}C=\frac{\det(II_2-\mathrm{i}II_1)}{\det I}\,\mathrm{d}S=2\rho_+\rho_-\,\mathrm{e}^{\phi}\,\mathrm{d}\xi_+\,\mathrm{d}\xi_-
\tag{28}
$$

where $\mathrm{d}S=\sqrt{-\det I}\,\mathrm{d}\xi_+\,\mathrm{d}\xi_-$ is the element of area of the string world sheet.

Let us prove the following statement: *for every solution* $\{\phi, \alpha_\pm, \rho_\pm\}$ *of the system* (26) *the functions* $g_\pm$ *and the real functions* $f_\pm$ *can be chosen in formulae* (27) *so that* $\forall(\xi^0, \xi^1)$

$$
K(\xi^0, \xi^1)\in SU(2).
\tag{29}
$$

Indeed, according to the definition of the functions ϕ, $\alpha_\pm$ the transformations (27) transforms the matrix $K(\xi^0, \xi^1)$ in the folowing way:

$$K \to \tilde{K} = G_- K G_+$$

where

$$G_+ = \begin{pmatrix} \exp(f_+/2) & 0 \\ g_+ & \exp(-f_+/2) \end{pmatrix} \qquad G_- = \begin{pmatrix} \exp(-f_-/2) & -g_- \\ 0 & \exp(f_-/2) \end{pmatrix}.$$

The general representation for matrix K as the solution of equation (9) is

$$K(\xi^0, \xi^1) = B_-(\xi_-) B_+(\xi_+)$$

where $B_\pm \in SL(2, C)$. Let functions $f_\pm$ be real functions and $g_\pm$ arbitrary complex functions. In accordance with the Iwasawa decomposition for the group $SL(2, C)$ we have

$$G_\pm B_\pm \in SU(2)$$

for some matrix $G_\pm$, and these matrices are unique. Consequently, condition (29) can be imposed.

Let us consider the transformation group (27) for the real functions $f_\pm$ only. The orbits of this transformation group decompose the set of string configurations of the model into non-intersecting classes. We declare different points of any orbit to be equivalent, so that in what follows we consider only the corresponding factor set. By virtue of this statement, within each orbit there exists a unique string configuration $(X_\mu, \Psi_\pm)$ such that the chiral field $K(\xi^0, \xi^1)$ constructed from it is an $SU(2)$-valued field. This means that the current $j_a(\xi)$ parametrizing the phase space of the internal degrees of freedom of the string can be assumed to be real in what follows. The transformation (27) with the pure imaginary functions $f_\pm$ from conformal invariance are also required.

Note that the transformation (27) can be rewritten in terms of the original string variables $(X_\mu, \Psi_\pm)$. This procedure probably provides a key for the construction of the full invariant action in our model.

5. Quantization

Our model was reduced to the product of the theory of a free relativistic particle (described by the variables P_μ, M_{mv}) and the theory of an $SU(2)$-valued chiral field $K(\xi^0, \xi^1)$. Non-triviality is conditioned by the constraints (15) and (22). In accordance with the Poisson brackets definition the resulting phase space is the product of the corresponding phase spaces. Thus we can use the method of Wightman (1964) which allows construction of 'new' quantum theories as derivatives of the 'old' ones.

From an algebraic viewpoint, the basic structure of the Hamiltonian description of our model is the algebra $A_1^{(1)} \otimes P$, where P is the Poincaré algebra and $A_1^{(1)} = (SL(2, C) \otimes [t^{-1}, t]) \otimes C_z$. We emphasize that the Heizenberg–Weyl algebra W_∞, which is traditional in the classical theory, is not needed in our following investigation. As seen, the object $A_1^{(1)} \otimes P$ is the most fundamental in the string theory. Thus, our classical studies lead to the following construction of the Hilbert space H of the quantum states:

$$H = H_P \otimes H_F \tag{30}$$

where $H_{\rm P}$ is the space of representation of the Poincaré algebra P and $H_{\rm F}$ is the corresponding space for the current algebra $A_1^{(1)}$. The representation of the current algebra will be constructed with the help of bosonization methods (Witten 1984). Let $H_{\rm F}$ be the Fock space of the fermionic (antisymmetrical) 2π-periodical wavefunctions and $a^\dagger_{\alpha,n}$, $a_{\beta,m}$ corresponding creation and annihilation operators:

$$[a^\dagger_{\alpha n}, a_{\beta m}]_+ = \delta_{\alpha\beta}\delta_{mn} \qquad \alpha, \beta = 1, 2; n, m = 0, \pm 1, \ldots .$$

The following pair of operator-valued functions are required by us:

$$\omega_1(\xi) = \sum_{n=0}^{\infty} a^\dagger_{1,n}\,\mathrm{e}^{-in\xi} + \sum_{n=1}^{\infty} a_{2,n}\,\mathrm{e}^{in\xi}$$

$$\omega_2(\xi) = \sum_{n=1}^{\infty} a^\dagger_{1,-n}\,\mathrm{e}^{-in\xi} + \sum_{n=0}^{\infty} a_{2,-n}\,\mathrm{e}^{in\xi}.$$

Let us define the quantum currents

$$J^a(\xi) = \sum_{\alpha,\beta} :\omega_\alpha(\xi)\sigma^a_{\alpha\beta}\omega_\beta(\xi):.$$

It is known that the following commutators occur:

$$[J_a(\xi), J_b(\eta)] = -2\mathrm{i}\delta_{ab}\delta'(\xi-\eta) - 2\mathrm{i}\varepsilon_{abc}J_c(\xi)\delta(\xi-\eta) \tag{31}$$

In accordance with the definitions of the classical current $j_a(\xi)$ and functions ϕ, π, $\alpha_\pm$, $\rho_\pm$, we have the following (nonlinear) functional dependence:

$$j_a(\xi) \equiv j_a[\phi(\xi), \pi(\xi), \alpha_\pm(\xi), \rho_\pm(\xi)].$$

Let us introduce the notation

$$j_{a\hbar} \equiv j_a[\hbar^{-1}\phi, \pi, \alpha_\pm, \hbar^{-1}\rho_\pm]$$

where $\hbar$ is the Planck constant. Note that we have the dependence

$$j_a(\xi) = F_a(\hbar; j_{a\hbar}(\xi))$$

which is trivial for the case $\hbar = 1$ only. Owing to one-to-one correspondence between the brackets (14) and (18) the brackets of the quantities $j_{a\hbar}(\xi)$ are

$$\{j_{a\hbar}(\xi), j_{b\hbar}(\eta)\} = -\frac{2}{\hbar}\,\delta_{ab}\delta'(\xi-\eta) - \frac{2}{\hbar}\,\varepsilon_{abc}j_{c\hbar}(\xi)\delta(\xi-\eta). \tag{32}$$

The quantization Υ of the internal degrees of freedom of our string is the correspondence

$$j_{a\hbar}(\xi) \to \Upsilon(j_{a\hbar}(\xi)) \equiv J_a(\xi). \tag{33}$$

Obviously, formulae (20) and (21) lead to the equality

$$[\Upsilon(j_{a\hbar}(\xi)), \Upsilon(j_{b\hbar}(\eta))] = \mathrm{i}\hbar\Upsilon(\{j_{a\hbar}(\xi), j_{b\hbar}(\eta)\}).$$

Let us discuss the Hilbert space $H_{\rm P}$. The representations of the Poincaré algebra P have been thoroughly investigated (Barut and Raczca 1987), and we choose the following structure of this space:

$$H_{\rm P} = \sum_s \int_{\mu^2>0} \mathrm{d}\mu^2 H_{\mu,s}.$$

The spaces $H_{\mu,s}$ are the spaces of the irreducible representation of the Poincaré algebra P which is marked by the eigenvalues μ^2 and $s(s+1)$ of the corresponding Casimir operators. For some basis $|n\rangle \equiv |\mu^2, s; n\rangle \in H_{\mu,s}$ we have $\langle m; l, \nu^2 | \mu^2, s; n\rangle = \delta(\mu^2 - \nu^2)\delta_{mn}\delta_{sl}$.

6. Physical subspace and the mass spectrum

Because of the presence of the first-type constraints (15) and (22), we have to select the physical states $|\phi_{\rm ph}\rangle$ by means of conditions

$$\langle \psi_{\rm ph} | (\mathbf{M} - 1) | \phi_{\rm ph}\rangle = 0 \tag{34}$$

$$\langle \psi_{\rm ph} | (P_\mu P^\mu)^2 \otimes (S_{(0)})^2 - S_\nu S^\nu \otimes (M_{(0)})^4 | \phi_{\rm ph}\rangle = 0 \tag{35}$$

where operators $\mathbf{M}$, $(\mathbf{S}_{(0)})^2 \equiv \mathbf{S}_{(0)\mu}\mathbf{S}^\mu_{(0)}$, $(\mathbf{M}_{(0)})^2 \equiv \mathbf{P}_{(0)\mu}\mathbf{P}^\mu_{(0)}$ are the operator functionals of operators $a^\dagger_{\alpha,n}$, $a_{\beta,m}$ only.

Let us investigate the physical vectors $|\phi_{\rm ph}\rangle$ in the form

$$|\phi_{\rm ph}\rangle = \sum_{m,n} \int_{\mu^2>0} \mathrm{d}\mu^2 v_{mn}(\mu; s) |\mu^2, s; m> \otimes |\varphi_n\rangle.$$

In this formula the vectors $|\varphi_n\rangle$ are some basis in the space $H_{\rm F}$. Because of the equalities

$$P_\nu P^\nu |\mu^2, s; n\rangle = \mu^2 |\mu^2, s; n\rangle \qquad S_\nu S^\nu |\mu, s; n\rangle = s(s+1) |\mu, s; n\rangle$$

condition (35) is transformed into the condition

$$\sum_{m,s,n,k} \int \mathrm{d}\mu^2 w^\dagger_{km}(\mu, s) v_{mn}(\mu, s) \{ s(s+1) \langle \varphi_k | (M_{(0)})^4 | \varphi_n\rangle$$
$$- \mu^4 \langle \varphi_k | (S_{(0)})^2 | \varphi_n\rangle \} = 0.$$

This is the equation for the functions $v_{mn}(\mu, s)$ and $w_{mk}(\mu, s)$. The following solution thus exists:

$$v_{mn}(\mu, s) = V_{mn}(s)\delta(\mu^2 - \mu_n^2) \qquad w_{mn} = W_{mn}(s)\delta(\mu^2 - \mu_n^2)$$

where

$$(\mu_n)^4 = s(s+1) \frac{\langle \varphi_n | (M_{(0)})^4 | \varphi_n\rangle}{\langle \varphi_n | (S_{(0)})^2 | \varphi_n\rangle}. \tag{36}$$

A full consideration of condition (34) will be left for future publications but some previous discussion will be considered. In accordance with formulae (13) the ξ^0 dynamics of the original string variables X_μ and $\Psi_\pm$ are realized through the matrices $T(\xi^1 + \xi^0)$ and $T(-\xi^1 + \xi^0)$. Therefore we have the following (classical) equation:

$$\mathrm{e}^{2\pi\{H,\ldots}T(\xi^0, \xi^1) = T(\xi^0, \xi^1)\mathbf{M}$$

where $\exp[\xi^0\{H, \ldots]$ is the classical 'operator' of the evolution

$$\exp[\xi^0\{H, \ldots]f = f + \xi^0\{H, f\} + \frac{(\xi^0)^2}{2}\{H, \{H, f\}\} + \ldots.$$

After quantization we have the quantum operator $\exp[i\xi^0 H]$. This means that we can impose the following condition instead of condition (34):

$$H|\varphi_n\rangle = m|\varphi_n\rangle \tag{37}$$

where $m = m(n) = 0, \pm 1, \ldots$.

Let us discuss the quantum Hamiltonian H. The following point must be taken into consideration. The introduction of quantum variables $\omega_i(\xi)$ without a natural classical image leads to some ambiguity for the Hamiltonian. Indeed, various cases of ξ^0 dynamics

$$\omega_i(0, \xi^1) \to \omega_i(\xi^0, \xi^1)\, e^{if(\xi^0)}$$

cannot be distinguishable in terms of the current $J^a(\xi^0, \xi^1)$ for various real functions $f(\xi^0)$. It is obvious for c-valued functions f, but more complicated cases are possible. That is why the quantum Hamiltonian H can be different from the Hamiltonian (24) which was subjected to the quantization (33). We choose H in the form

$$H = \sum_n |n|\,(a_{1,n}^\dagger a_{1,n} + a_{2,n}^\dagger a_{2,n}).$$

A similar form is well known for Hamiltonians in models with bosonization (Green *et al* 1987). Obviously, the eigenvectors for the spectral problem (37) can be found explicitly.

7. Concluding remarks

As seen, the relativistic string is one of the simple models for a free particle with an infinite number of internal degrees of freedom. In this connection, the suggested approach is to generalize the Wigner description of a free elementary particle as some operator representation of the Poincaré algebra P. The quantum string in our work is the representation of the object $A_1^{(1)} \otimes P$. The non-triviality of the mass spectrum is the consequence of the 'kinematic' constraint (22). As seen, we have the Regge trajectories which are nonlinear but linear asymptotically:

$$s = -\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \alpha_\varphi^2 \mu_\varphi^4}$$

where

$$\alpha_\varphi^2 \equiv \frac{\langle\varphi|(S_0)^2|\varphi\rangle}{\langle\varphi|(M_0)^4|\varphi\rangle} \quad - \quad \text{for some } |\varphi\rangle \in H_F.$$

The development of this idea probably requires some selection rules for vectors $|\varphi\rangle$. The author hopes to return to this question in subsequent studies.

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